

WEYL DENOMINATOR IDENTITY FOR THE AFFINE LIE SUPERALGEBRA $\mathfrak{gl}(2|2)$

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ABSTRACT. We prove the Weyl denominator identity for the affine Lie superalgebra $\mathfrak{gl}(2|2)$ conjectured by V. Kac and M. Wakimoto in [KW]. As it was pointed out in [KW], this gives a new proof of the Jacobi identity for the number of presentations of a given integer as a sum of 8 squares.

0. INTRODUCTION

The denominator identities for Lie superalgebras were formulated and partially proven in the paper of V. Kac and M. Wakimoto [KW]. In the same paper it was shown how various classical identities in number theory as the number of representation of a given integer as a sum of d squares can be obtained, for some d , by evaluation of certain denominator identities. The following cases are considered in the paper [KW]:

- (a) basic Lie superalgebras, i.e. the finite-dimensional simple Lie superalgebras, which have a reductive even part and admit an even non-degenerate invariant bilinear form;
- (b) the affinization of basic Lie superalgebras with non-zero dual Coxeter number;
- (c) the (twisted) affinization of a strange Lie superalgebras $Q(n)$;
- (d) the affinization of $\mathfrak{gl}(2|2)$ (this is the smallest basic Lie superalgebras with zero dual Coxeter number).

Some of the cases (a), (b) are proven in [KW]; the proof is based on combinatorics of root systems and a certain result from representation theory. The rest of (a) was proven in [G1] using only combinatorics of root systems. The rest of (b) was proven in [G2] using (a) and the existence of Casimir operator. The case (c) was proven in [Z] analytically. In the present paper we prove (d), i.e. the identity for the affine Lie superalgebra $\mathfrak{gl}(2|2)$ conjectured in [KW], 7.1. The proof uses the existence of Casimir operator and an idea of [Z].

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In order to write down the identity, we introduce the following infinite products after [DK]: for a parameter q and a formal variable x we set

$$(1+x)_q^\infty := \prod_{n=0}^{\infty} (1+q^n x), \quad \text{and} \quad (1-x)_q^\infty := \prod_{n=0}^{\infty} (1-q^n x).$$

These infinite products converge for any $x \in \mathbb{C}$ if the parameter q is a real number $0 < q < 1$. In particular, they are well-defined for $0 < x = q < 1$ and $(1 \pm q)_q^\infty := \prod_{n=1}^{\infty} (1 \pm q^n)$.

Take the formal variables x, y_1, y_2 . The denominator identity for $\mathfrak{gl}(2|2)$ can be written in the following form

(1)

$$\begin{aligned} & \frac{(1-x)_q^\infty (1-qx^{-1})_q^\infty (1-xy_1y_2)_q^\infty (1-q(xy_1y_2)^{-1})_q^\infty ((1-q)_q^\infty)^4}{\prod_{i=1}^2 (1+y_i)_q^\infty (1+qy_i^{-1})_q^\infty (1+xy_i)_q^\infty (1+qx^{-1}y_i^{-1})_q^\infty} = \\ & = \frac{((1-q)_q^\infty)^2}{(1-qy_1^{-1}y_2)_q^\infty (1-qy_1y_2^{-1})_q^\infty} \cdot \sum_{n=-\infty}^{\infty} \left(\frac{q^n}{(1+q^n y_1)(1+q^n y_2)} - \frac{q^n x}{(1+q^n x y_1)(1+q^n x y_2)} \right). \end{aligned}$$

Expanding the factor $\frac{((1-q)_q^\infty)^2}{(1-qy_1^{-1}y_2)_q^\infty (1-qy_1y_2^{-1})_q^\infty}$ in the region $q < |\frac{y_1}{y_2}| < q^{-1}$ we obtain (see Lemma 1.3.1)

$$\begin{aligned} & \frac{((1-q)_q^\infty)^2}{(1-qy_1^{-1}y_2)_q^\infty (1-qy_1y_2^{-1})_q^\infty} = 1 + \sum_{n=1}^{\infty} f_n\left(\frac{y_1}{y_2}\right), \\ & \text{where } f_n(y) := (y^n + y^{-n} - y^{n-1} - y^{1-n}) \sum_{j=0}^{\infty} (-1)^j q^{(j+1)(j+2n)/2} \end{aligned}$$

and this gives the identity conjectured by V. Kac and M. Wakimoto.

The left-hand side of the identity represents the Weyl denominator \hat{R} for the affine Lie superalgebra $\mathfrak{gl}(2|2)$; the second factor in the right-hand side is the analogue of the right-hand side of the denominator identity for affine Lie superalgebras with non-zero dual Coxeter number. Note that the denominator identity for the affine Lie superalgebra $\mathfrak{sl}(2|2)$ can be obtained from the denominator identity for $\mathfrak{gl}(2|2)$ by taking $y_1 = y_2$; as a result, the denominator identity for $\mathfrak{sl}(2|2)$ is almost similar to the denominator identity for affine Lie superalgebras with non-zero dual Coxeter number with one extra-factor $(1-q)_q^\infty$ in the left-hand side (since the dimension of Cartan subalgebra for $\mathfrak{sl}(2|2)$ is less by one than the dimension of Cartan subalgebra for $\mathfrak{gl}(2|2)$).

As it is shown in [KW], the evaluation of this identity gives the following Jacobi identity [J]:

$$(2) \quad \square(q)^8 = 1 + 16 \sum_{j,k=1}^{\infty} (-1)^{(j+1)k} k^3 q^{jk},$$

where $\square(q) = \sum_{j \in \mathbb{Z}} q^{j^2}$ and thus the coefficient of q^m in the power series expansion of $\square(q)^d$ is the number of representation of a given integer as a sum of d squares (taking into the account the order of summands).

In Section 1 we introduce notation. In Section 2 we prove the identity (1). In Section 3 we recall how to deduce the Jacobi identity from the identity (1).

1. NOTATION

1.1. Root system. Consider $V := \mathbb{R}^5$ endowed by a bilinear form and an orthogonal basis $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2, \delta$ such that $(\varepsilon_i, \varepsilon_i) = 1 = -(\delta_i, \delta_i)$ for $i = 1, 2$ and $(\delta, \delta) = 0$. Set

$$\beta_1 := \delta_1 - \varepsilon_1, \quad \alpha := \varepsilon_1 - \varepsilon_2, \quad \beta_2 := \varepsilon_2 - \delta_2, \quad \gamma := \beta_1 + \alpha + \beta_2 = \delta_1 - \delta_2.$$

The root system of $\mathfrak{gl}(2|2)$ is $\Delta_0 = \{\pm\alpha, \pm\gamma\}$, $\Delta_1 = \{\pm\beta_i; \pm(\alpha + \beta_i)\}_{i=1,2}$. The affine root system is $\hat{\Delta}_i = \cup_{s \in \mathbb{Z}} (\Delta_i + s\delta)$, $i = 0, 1$.

We consider the following sets of simple roots for $\mathfrak{gl}(2|2)$ and $\mathfrak{gl}(2|2)^\wedge$ respectively:

$$\Pi := \{\beta_1, \alpha, \beta_2\}, \quad \text{and} \quad \hat{\Pi} = \{\beta_1, \alpha, \beta_2, \delta - \gamma\}.$$

One has

$$\Delta_+ = \{\alpha, \gamma; \beta_i, \alpha + \beta_i\}_{i=1,2}, \quad \hat{\Delta}_+ = \Delta_+ \cup \cup_{s=1}^{\infty} (\Delta + s\delta), \quad \hat{\rho} = \rho = -\frac{\beta_1 + \beta_2}{2}.$$

Set

$$Q^+ := \sum_{\mu \in \Pi} \mathbb{Z}_{\geq 0} \mu, \quad \hat{Q}^+ = \sum_{\mu \in \hat{\Pi}} \mathbb{Z}_{\geq 0} \mu.$$

1.1.1. For $\nu \in \Delta_0$ let $s_\nu \in \text{Aut}(V)$ be the reflection with respect to ν , i.e. $s_\nu(\lambda) = \lambda - \frac{(\lambda, \nu)}{(\nu, \nu)} \nu$. The Weyl group W of Δ_0 takes form $W = W_\alpha \times W_\gamma$, where W_α (resp., W_γ) is generated by the reflection s_α (resp., s_γ).

For $\nu \in V$ introduce $t_\nu \in \text{Aut}(V)$ by the formula

$$t_\mu(\lambda) = \lambda - (\lambda, \mu) \delta.$$

Then $t_\mu t_\nu = t_{\mu+\nu}$. For $\nu \in \Delta_0$ we denote by T_ν the infinite cyclic group generated by t_ν and by \hat{W}_ν the group generated by s_ν and t_ν . The Weyl group of $\mathfrak{gl}(2|2)^\wedge$ is $\hat{W} = \hat{W}_\alpha \times \hat{W}_\gamma$. Notice that δ and $\beta_1 - \beta_2$ lie in the kernel of the bilinear form so these vectors are \hat{W} -stable.

For a subgroup G of the Weyl group we introduce the following operator:

$$\mathcal{F}_G := \sum_{w \in G} \text{sgn } w \cdot w.$$

1.2. Algebra \mathcal{R} . We are going to use notation of [G2], 1.4, which we recall below.

1.2.1. Consider the space $\hat{\mathfrak{h}}^* = V \oplus \mathbb{R}\Lambda_0$ and extend our bilinear form by $(\Lambda_0, \delta) = 1$, $(\Lambda_0, \Lambda_0) = (\Lambda_0, \varepsilon_i) = (\Lambda_0, \delta_i) = 0$ for $i = 1, 2$. The Weyl group \hat{W} acts on $\hat{\mathfrak{h}}^*$ as follows: the reflections act by the same formulas and the action of t_μ extends by the standard formula

$$t_\mu(\lambda) = \lambda + (\lambda, \delta)\mu - ((\lambda, \mu) + \frac{(\mu, \mu)}{2}(\lambda, \delta))\delta, \quad \mu \in V, \lambda \in \hat{\mathfrak{h}}^*$$

Call a \hat{Q}^+ -cone a set of the form $(\lambda - \hat{Q}^+)$, where $\lambda \in \hat{\mathfrak{h}}^*$.

1.2.2. For a formal sum of the form $Y := \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu$, $b_\nu \in \mathbb{Q}$ define the *support* of Y by $\text{supp}(Y) := \{\nu \in \hat{\mathfrak{h}}^* \mid b_\nu \neq 0\}$. Let \mathcal{R} be a vector space over \mathbb{Q} , spanned by the sums of the form $\sum_{\nu \in \hat{Q}^+} b_\nu e^{\lambda - \nu}$, where $\lambda \in \hat{\mathfrak{h}}^*$, $b_\nu \in \mathbb{Q}$. In other words, \mathcal{R} consists of the formal sums $Y = \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu$ with the support lying in a finite union of \hat{Q}^+ -cones.

Clearly, \mathcal{R} has a structure of commutative algebra over \mathbb{Q} . If $Y \in \mathcal{R}$ is such that $YY' = 1$ for some $Y' \in \mathcal{R}$, we write $Y^{-1} := Y'$.

1.2.3. *Action of the Weyl group.* For $w \in \hat{W}$ set $w(\sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu) := \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^{w\nu}$. One has $wY \in \mathcal{R}$ iff $w(\text{supp } Y)$ is a subset of a finite union of \hat{Q}^+ -cones.

Let W' be a subgroup of \hat{W} . Let $\mathcal{R}_{W'} := \{Y \in \mathcal{R} \mid wY \in \mathcal{R} \text{ for each } w \in W'\}$. Clearly, $\mathcal{R}_{W'}$ is a subalgebra of \mathcal{R} .

1.2.4. *Infinite products.* An infinite product of the form $Y = \prod_{\nu \in X} (1 + a_\nu e^{-\nu})^{r(\nu)}$, where $a_\nu \in \mathbb{Q}$, $r(\nu) \in \mathbb{Z}_{\geq 0}$ and $X \subset \hat{\Delta}$ is such that the set $X \setminus \hat{\Delta}_+$ is finite, can be naturally viewed as an element of \mathcal{R} ; clearly, this element does not depend on the order of factors. Let \mathcal{Y} be the set of such infinite products. For any $w \in \hat{W}$ the infinite product

$$wY := \prod_{\nu \in X} (1 + a_\nu e^{-w\nu})^{r(\nu)},$$

is again an infinite product of the above form, since, as one easily sees ([G2], Lem. 1.2.8), the set $w\hat{\Delta}_+ \setminus \hat{\Delta}_+$ is finite. Hence \mathcal{Y} is a \hat{W} -invariant multiplicative subset of $\mathcal{R}_{\hat{W}}$.

The elements of \mathcal{Y} are invertible in \mathcal{R} : using the geometric series we can expand Y^{-1} (for example, $(1 - e^\alpha)^{-1} = -e^{-\alpha}(1 - e^{-\alpha})^{-1} = -\sum_{i=1}^{\infty} e^{-i\alpha}$).

1.2.5. *The subalgebra \mathcal{R}' .* Denote by \mathcal{R}' the localization of $\mathcal{R}_{\hat{W}}$ by \mathcal{Y} . By above, \mathcal{R}' is a subalgebra of \mathcal{R} . Observe that $\mathcal{R}' \not\subset \mathcal{R}_{\hat{W}}$: for example, $(1 - e^{-\alpha})^{-1} \in \mathcal{R}'$, but $(1 - e^{-\alpha})^{-1} = \sum_{j=0}^{\infty} e^{-j\alpha} \notin \mathcal{R}_{\hat{W}}$. We extend the action of \hat{W} from $\mathcal{R}_{\hat{W}}$ to \mathcal{R}' by setting $w(Y^{-1}Y') := (wY)^{-1}(wY')$ for $y \in \mathcal{Y}$, $Y' \in \mathcal{R}_{\hat{W}}$.

An infinite product of the form $Y = \prod_{\nu \in X} (1 + a_\nu e^{-\nu})^{r(\nu)}$, where a_ν, X are as above and $r(\nu) \in \mathbb{Z}$ lies in \mathcal{R}' , and $wY = \prod_{\nu \in X} (1 + a_\nu e^{-w\nu})^{r(\nu)}$. One has

$$\text{supp}(Y) \subset \lambda' - \hat{Q}^+, \quad \text{where } \lambda' := - \sum_{\nu \in X \setminus \hat{\Delta}_+ : a_\nu \neq 0} r_\nu \nu.$$

Remark. Set $q := e^{-\delta}, x := e^{-\alpha}, y_i := e^{-\beta_i}$ and write elements of \mathcal{R}' as power series in these variables. Since $\{e^{-\nu}, \nu \in \hat{\Pi}\} = \{x, y_1, y_2, q(xy_1y_2)^{-1}\}$, the support of $Y \in \mathcal{R}'$ correspond to the expansion of Y in the region $|q| < |xy_1y_2|; |x|, |y_1|, |y_2| < 1$.

1.2.6. Let W' be a subgroup of \hat{W} . For $Y \in \mathcal{R}'$ we say that Y is W' -invariant (resp., W' -anti-invariant) if $wY = Y$ (resp., $wY = \text{sgn}(w)Y$) for each $w \in W'$.

Let $Y = \sum a_\mu e^\mu \in \mathcal{R}_{W'}$ be W' -anti-invariant. Then $a_{w\mu} = (-1)^{\text{sgn}(w)} a_\mu$ for each μ and $w \in W'$. In particular, $W' \text{supp}(Y) = \text{supp}(Y)$, and, moreover, for each $\mu \in \text{supp}(Y)$ one has $\text{Stab}_{W'} \mu \subset \{w \in W' \mid \text{sgn}(w) = 1\}$. The condition $Y \in \mathcal{R}_{W'}$ is essential: for example, for $W' = \{\text{id}, s_\alpha\}$, the expressions $Y := e^\alpha - e^{-\alpha}$, $Y^{-1} = e^{-\alpha}(1 - e^{-2\alpha})^{-1}$ are W' -anti-invariant, but $\text{supp}(Y^{-1}) = -\alpha, -3\alpha, \dots$ is not s_α -invariant.

Take $Y = \sum a_\mu e^\mu \in \mathcal{R}_{W'}$. The sum $\mathcal{F}_{W'}(Y) = \sum_{w \in W'} \text{sgn}(w)wY$ is an element of \mathcal{R} if for each μ the sum $\sum_{w \in W'} \text{sgn}(w)a_{w\mu}$ is finite (i.e., $W'\mu \cap \text{supp}(Y)$ is finite). In this case $\mathcal{F}_{W'}(Y) \in \mathcal{R}$ and, writing $\mathcal{F}_{W'}(Y) = \sum b_\mu e^\mu$, we obtain $b_\mu = \sum_{w \in W'} \text{sgn}(w)a_{w\mu}$ so $b_\mu = \text{sgn}(w)b_{w\mu}$ for each $w \in W'$. We conclude that

$$Y \in \mathcal{R}_{W'} \ \& \ \mathcal{F}_{W'}(Y) \in \mathcal{R} \implies \begin{cases} \mathcal{F}_{W'}(Y) \in \mathcal{R}_{W'}; \\ \mathcal{F}_{W'}(Y) \text{ is } W'\text{-anti-invariant}; \\ \text{supp}(\mathcal{F}_{W'}(Y)) \text{ is } W'\text{-stable.} \end{cases}$$

Let us call a vector $\lambda \in \hat{\mathfrak{h}}^*$ W' -regular if $\text{Stab}_{W'} \lambda = \{\text{id}\}$. Say that the orbit $W'\lambda$ is W' -regular if λ is W' -regular (so the orbit consists of W' -regular points). If W' is an affine Weyl group, then for any $\lambda \in \hat{\mathfrak{h}}^*$ the stabilizer $\text{Stab}_{W'} \lambda$ is either trivial or contains a reflection. Thus for $W' = \hat{W}_\alpha, \hat{W}_\gamma$ one has

$$Y \in \mathcal{R}_{W'} \ \& \ \mathcal{F}_{W'}(Y) \in \mathcal{R} \implies \text{supp}(\mathcal{F}_{W'}(Y)) \text{ is a union of } W'\text{-regular orbits.}$$

1.2.7. *Remark.* For $Y \in \mathcal{R}'$ the sum $\mathcal{F}_{W'}(Y)$ is not always W' -anti-invariant: for example, for $W' = \{\text{id}, s_\alpha\}$ one has $\mathcal{F}_{W'}((1 - e^{-\alpha})^{-1}) = (1 - e^{-\alpha})^{-1} - (1 - e^\alpha)^{-1} = 1 + 2e^{-\alpha} + 2e^{-2\alpha} + \dots$ which is not W' -anti-invariant.

1.3. **Another form of denominator identity.** Introduce the following elements of \mathcal{R} :

$$\begin{aligned} R_0 &:= \prod_{\nu \in \Delta_{0,+}} (1 - e^{-\nu}), & R_1 &:= \prod_{\nu \in \Delta_{1,+}} (1 + e^{-\nu}), & R &:= \frac{R_0}{R_1}, \\ \hat{R}_0 &:= \prod_{\nu \in \hat{\Delta}_{0,+}} (1 - e^{-\nu}), & \hat{R}_1 &:= \prod_{\nu \in \hat{\Delta}_{1,+}} (1 + e^{-\nu}), & \hat{R} &:= \frac{\hat{R}_0}{\hat{R}_1}. \end{aligned}$$

The products Re^ρ and $\hat{R}e^\rho$ are \hat{W} -anti-invariant elements of \mathcal{R}' (see, for instance, [G2], Lem. 1.5.1).

1.3.1. **Lemma.** *In the region $q < |y| < q^{-1}$ one has*

$$\frac{((1-q)_q^\infty)^2}{(1-qq)_q^\infty(1-qq^{-1})_q^\infty} = \sum_{n=1}^{\infty} (y^n + y^{-n} - y^{n-1} - y^{1-n}) \sum_{j=0}^{\infty} (-1)^j q^{(j+1)(j+2n)/2}$$

and this expression lies in \mathcal{R} for $y = e^{\beta_2 - \beta_1}$.

Proof. Consider the root system $\mathfrak{sl}(2|1)$ with the odd simple roots β'_1, β'_2 and the even positive root $\alpha' = \beta'_1 + \beta'_2$. Note that the corresponding element ρ' is equal to zero. Consider the corresponding affine root system, let δ' be the minimal imaginary root and \hat{W}' be its Weyl group. The affine denominator identity for $\mathfrak{sl}(2|1)$ written for $z := e^{-\delta}$ takes form

$$\frac{(1 - e^{-\alpha'})_z^\infty (1 - ze^{\alpha'})_z^\infty ((1 - z)_z^\infty)^2}{\prod_{i=1}^2 (1 + e^{-\beta_i})_z^\infty (1 + ze^{\beta_i})_z^\infty} = \sum_{n=-\infty}^{\infty} z^{n^2} \left(\frac{e^{n\alpha}}{1 + z^n e^{-\beta_1}} - \frac{e^{-n\alpha}}{1 + z^n e^{\beta_2}} \right).$$

Both sides are well-defined for real $0 < z < 1$ and β'_i such that $e^{\beta'_i} \neq z^n$ for $n \in \mathbb{Z}$. Taking $e^{\alpha'} = -1$ and $e^{-\beta'_1} := -\xi$ we obtain $e^{-\beta'_2} = e^{-\alpha} e^{\beta'_1} = \xi^{-1}$ and the evaluation gives

$$\frac{2((1+z)_z^\infty (1-z)_z^\infty)^2}{(1-\xi)_z^\infty (1+\xi^{-1})_z^\infty (1-z\xi^{-1})_z^\infty (1+z\xi)_z^\infty} = \sum_{n=-\infty}^{\infty} (-1)^n z^{n^2} \left(\frac{1}{1 - z^n \xi} - \frac{1}{1 + z^n \xi} \right).$$

For $z^2 = q, \xi^2 = y$ we get

$$\frac{((1-q)_q^\infty)^2}{(1-qq)_q^\infty (1-qq^{-1})_q^\infty (1-y)} = 2 \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{m^2+m}{2}} \frac{1}{1 - q^m y}.$$

For $m > 0$ one has $\frac{1}{1-q^m y} = \sum_{k=0}^{\infty} q^{mk} y^k$ and $\frac{1}{1-q^{-m} y} = -\sum_{k=1}^{\infty} q^{mk} y^{-k}$ so

$$\begin{aligned} \frac{((1-q)_q^\infty)^2}{(1-qq)_q^\infty (1-qq^{-1})_q^\infty} &= 1 + (1-y) \sum_{m=1}^{\infty} (-1)^m \sum_{k=0}^{\infty} (q^{\frac{m^2+m+2mk}{2}} y^k - q^{\frac{m^2-m+2mk}{2}} y^{-k}) \\ &= 1 + \sum_{m=0}^{\infty} (-1)^m \sum_{k=1}^{\infty} (q^{\frac{(m+1)(m+2k)}{2}} (y^k - y^{k-1} + y^{-k} - y^{1-k})) \end{aligned}$$

as required. One readily sees that the right-hand side of the above expression lies in \mathcal{R} for $y = e^{\beta_2 - \beta_1}$. \square

1.3.2. Set $x := e^{-\alpha}, y_i := e^{-\beta_i}$ for $i = 1, 2$ and $q := e^{-\delta}$. Under this substitution, the left-hand side of (1) becomes \hat{R} and, using Lemma 1.3.1, we rewrite (1) in the following form

$$(3) \quad \hat{R}e^\rho = (1 + \sum f_n) e^{-\rho} \mathcal{F}_{\hat{W}_\alpha} \left(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right).$$

Denote by *LHS* (resp., *RHS*) the left-hand (resp., right-hand) side of the identity (3).

The denominator identity for $\mathfrak{gl}(2|2)$ takes the form

$$(4) \quad \mathcal{F}_{W_\alpha} \left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right) = Re^\rho = \mathcal{F}_{W_\gamma} \left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right).$$

so (3) can be rewritten as

$$\hat{R}e^\rho = (1 + \sum f_n) \mathcal{F}_{T_\alpha}(Re^\rho).$$

In the sequel we need the following lemma.

1.3.3. Lemma. *If $\mathcal{F}_{T_\alpha}(Re^\rho)$ is well-defined (as an element of \mathcal{R}), then*

$$\mathcal{F}_{T_\alpha}(Re^\rho) = \mathcal{F}_{T_\gamma}(Re^\rho).$$

Proof. Note that $(\gamma - \alpha, \rho) = (\gamma - \alpha, \beta_i) = 0$ for $i = 1, 2$ so $\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}$ is invariant with respect to the action of $t_{\gamma-\alpha}$. Therefore

$$\mathcal{F}_{T_\alpha} \left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right) = \mathcal{F}_{T_\gamma} \left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right).$$

Using the formula (4), we obtain

$$\begin{aligned} \mathcal{F}_{T_\alpha}(Re^\rho) &= \mathcal{F}_{T_\alpha} \circ \mathcal{F}_{W_\gamma} \left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right) = \mathcal{F}_{W_\gamma} \circ \mathcal{F}_{T_\alpha} \left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right) \\ &= \mathcal{F}_{W_\gamma} \circ \mathcal{F}_{T_\gamma} \left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right) = \mathcal{F}_{T_\gamma} \circ \mathcal{F}_{W_\gamma} \left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right) = \mathcal{F}_{T_\gamma}(Re^\rho), \end{aligned}$$

as required. \square

As a corollary, (3) can be rewritten as $\hat{R}e^\rho = (1 + \sum f_n) \mathcal{F}_{T_\gamma}(Re^\rho)$.

2. PROOF OF THE DENOMINATOR IDENTITY

2.1. By 1.2.4, LHS is an invertible element of \mathcal{R}' . In this subsection we show that RHS is a well-defined element of \mathcal{R} .

For $w \in \hat{W}_\alpha$ set

$$S_w := \text{supp} \left(w \left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right) \right).$$

One has

$$t_{n\alpha} \left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right) = \frac{e^\rho q^n}{(1+q^n e^{-\beta_1})(1+q^n e^{-\beta_2})}.$$

Then for $n > 0$ one has

$$(5) \quad \begin{aligned} S_{\text{id}} &= \{\rho - k_1\beta_1 - k_2\beta_2\}, & S_{s_\alpha} &= \{\rho - (1 + k_1 + k_2)\alpha - k_1\beta_1 - k_2\beta_2\}, \\ S_{t_{n\alpha}} &= \{\rho - n(1 + k_1 + k_2)\delta - k_1\beta_1 - k_2\beta_2\}, \\ S_{t_{-n\alpha}} &= \{\rho - n(1 + k_1 + k_2)\delta + (k_1 + 1)\beta_1 + (k_2 + 1)\beta_2\}, \\ S_{t_{n\alpha}s_\alpha} &= \{\rho - n(1 + k_1 + k_2)\delta - (1 + k_1 + k_2)\alpha - k_1\beta_1 - k_2\beta_2\}, \\ S_{t_{-n\alpha}s_\alpha} &= \{\rho - n(1 + k_1 + k_2)\delta + (1 + k_1 + k_2)\alpha + k_1\beta_1 + k_2\beta_2\}, \end{aligned}$$

where $k_1, k_2 \geq 0$. Observe that the above sets are pairwise disjoint so the sum $\mathcal{F}_{\hat{W}_\alpha} \left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right)$ is well-defined and its support lies in $\rho - \hat{Q}^+$. Clearly, the sum $1 + \sum f_n$ is well-defined.

One readily sees that

$$(6) \quad n\delta + k\beta_1 - k\beta_2 \in \hat{Q}^+ \iff |k| \leq n$$

Thus the support of $1 + \sum f_n$ lies in $\{0\} \cup \{-m\delta + k\beta_1 - k\beta_2 \mid m > 0\} \cap -\hat{Q}^+$ (in particular, $(1 + \sum f_n) \in \mathcal{R}$). Hence RHS is a well-defined element of \mathcal{R} .

2.2. Lemma. *The expansion of $\frac{RHS}{LHS}$ in the region $|q| < |xy_1y_2|, |x|, |y_1|, |y_2| < 1$ is of the form $1 + \sum_{n=1}^{\infty} \sum_{j=-n}^n a_{n,j} q^n \left(\frac{y_1}{y_2}\right)^j$, where $a_{n,j} \in \mathbb{Z}$.*

Proof. Recall that $LHS = \hat{R}e^\rho$ and that $\hat{R} \in \mathcal{Y}$ (see 1.2.4 for notation). By 2.1, $RHS \in \mathcal{R}$. Therefore the fraction

$$Y := \frac{RHS}{LHS} = \hat{R}^{-1}e^{-\rho} \cdot RHS$$

lies in \mathcal{R} .

Clearly, $-\rho \in \text{supp}(\hat{R}^{-1}e^{-\rho}) \subset (-\rho - \hat{Q}^+)$. Since $\text{supp}(RHS) \subset \rho - \hat{Q}^+$, we conclude that $\text{supp}(Y) \subset -\hat{Q}^+$. By (5) the coefficient of e^ρ in RHS is 1; clearly, the coefficient of $e^{-\rho}$ in $\hat{R}^{-1}e^{-\rho}$ is also 1, so the coefficient of $e^0 = 1$ in Y is 1. In the light of Remark 1.2.5, the required assertion is equivalent to the inclusion

$$(7) \quad \text{supp}(Y) \subset \{-n\delta + j(\beta_1 - \beta_2) \mid n \geq 0, |j| \leq n\}.$$

Retain notation of 1.2.1. The element $\hat{\rho}_\alpha := 2\Lambda_0 + \frac{\alpha}{2}$ is the standard element for the corresponding copy of $\mathfrak{sl}_2 \subset \mathfrak{gl}(2|2)$. Recall that $\hat{R} = \frac{\hat{R}_0}{\hat{R}_1}$ (see 1.3 for notation) so $\hat{R}_1 e^{\hat{\rho}_\alpha - \rho} = \hat{R}_0 e^{\hat{\rho}_\alpha} \cdot (\hat{R}e^\rho)^{-1}$. By 1.2.4, $\hat{R}_1 e^{\hat{\rho}_\alpha - \rho}$ belongs to $\mathcal{R}_{\hat{W}}$. It is a standard fact that $\hat{R}_0 e^{\hat{\rho}_\alpha}$ is \hat{W}_α -anti-invariant. Recall that $\hat{R}e^\rho$ is \hat{W} -anti-invariant. Thus $\hat{R}_1 e^{\hat{\rho}_\alpha - \rho}$ is a \hat{W}_α -invariant element of $\mathcal{R}_{\hat{W}}$. One has

$$\hat{R}_0 e^{\hat{\rho}_\alpha} Y = \hat{R}_1 e^{\hat{\rho}_\alpha - \rho} \cdot RHS = (1 + \sum f_n) \cdot \hat{R}_1 e^{\hat{\rho}_\alpha - \rho} \cdot \mathcal{F}_{\hat{W}_\alpha} \left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})} \right).$$

The \hat{W}_α -invariance of $\hat{R}_1 e^{\hat{\rho}_\alpha - \rho}$ gives

$$\hat{R}_1 e^{\hat{\rho}_\alpha - \rho} \cdot \mathcal{F}_{\hat{W}_\alpha} \left(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right) = \mathcal{F}_{\hat{W}_\alpha} \left(\frac{\hat{R}_1 e^{\hat{\rho}_\alpha}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right)$$

so

$$\hat{R}_0 e^{\hat{\rho}_\alpha} Y = (1 + \sum f_n) \cdot \mathcal{F}_{\hat{W}_\alpha} \left(\frac{\hat{R}_1 e^{\hat{\rho}_\alpha}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right).$$

By 2.1, $\mathcal{F}_{\hat{W}_\alpha} \left(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right)$ lies in \mathcal{R} so $\mathcal{F}_{\hat{W}_\alpha} \left(\frac{\hat{R}_1 e^{\hat{\rho}_\alpha}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right)$ lies in \mathcal{R} . By 1.2.4, the term

$$\frac{\hat{R}_1 e^{\hat{\rho}_\alpha}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} = e^{\hat{\rho}_\alpha} \prod_{\beta \in \hat{\Delta}_{1,+} \setminus \{\beta_1, \beta_2\}} (1 + e^{-\beta})$$

lies in $\mathcal{R}_{\hat{W}}$. Therefore, in the light of 1.2.6, $\mathcal{F}_{\hat{W}_\alpha} \left(\frac{\hat{R}_1 e^{\hat{\rho}_\alpha}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right)$ is a \hat{W}_α -anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. Observe that $\frac{y_1}{y_2} = e^{\beta_2 - \beta_1}$ is \hat{W} -invariant so f_n is \hat{W} -invariant. Thus $(1 + \sum f_n) \mathcal{F}_{\hat{W}_\alpha} \left(\frac{\hat{R}_1 e^{\hat{\rho}_\alpha}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right)$ is a \hat{W}_α -anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. As a result, $\hat{R}_0 e^{\hat{\rho}_\alpha} Y$ is a \hat{W}_α -anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$.

Write $Y = Y_1 + Y_2$, where $Y_1 = \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} a_{n,j} q^n \left(\frac{y_1}{y_2} \right)^j$ and Y_2 does not have monomials of the form $q^n \left(\frac{y_1}{y_2} \right)^j$, i.e $\text{supp}(Y) = \text{supp}(Y_1) \amalg \text{supp}(Y_2)$. One has $Y_i \in \mathcal{R}$ because $\text{supp}(Y_i) \subset \text{supp}(Y) \subset -\hat{Q}^+$ ($i = 1, 2$).

Since $\frac{y_1}{y_2} = e^{\beta_2 - \beta_1}$ is \hat{W} -invariant, Y_1 is a \hat{W} -invariant element of $\mathcal{R}_{\hat{W}}$. Since $\hat{R}_0 e^{\hat{\rho}_\alpha}$ is a \hat{W}_α -anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$, the product $\hat{R}_0 e^{\hat{\rho}_\alpha} Y_1$ is a \hat{W}_α -anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. By above, $\hat{R}_0 e^{\hat{\rho}_\alpha} Y$ is a \hat{W}_α -anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. Hence $\hat{R}_0 e^{\hat{\rho}_\alpha} Y_2$ is a \hat{W}_α -anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$.

Assume that $Y_2 \neq 0$. Recall that $\text{supp}(Y_2) \subset -\hat{Q}^+$. Let μ be a maximal element in $\text{supp}(Y_2)$ with respect to the standard partial order $\mu \leq \nu$ if $(\nu - \mu) \in \hat{Q}^+$. Then $\hat{\rho}_\alpha + \mu$ is a maximal element in the support of $\hat{R}_0 e^{\hat{\rho}_\alpha} Y_2$. By 1.2.6, this support is the union of \hat{W}_α -regular orbits, so $\hat{\rho}_\alpha + \mu$ is a maximal element in a regular \hat{W}_α -orbit (regularity means that each element has the trivial stabilizer in \hat{W}_α). Since $\mu \in -\hat{Q}^+$ one has $\frac{2(\mu, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$. Therefore $\frac{2(\hat{\rho}_\alpha + \mu, \alpha)}{(\alpha, \alpha)} = 1 + (\mu, \alpha)$, $\frac{2(\hat{\rho}_\alpha + \mu, \delta - \alpha)}{(\delta - \alpha, \delta - \alpha)} = 1 + (\mu, \delta - \alpha)$ are positive integers so $(\mu, \alpha), (\mu, \delta - \alpha) \geq 0$. Since $\mu \in -\hat{Q}^+$ one has $(\mu, \delta) = 0$ and thus $(\mu, \alpha) = 0$.

The element $\hat{\rho}_\gamma := -2\Lambda_0 + \frac{\gamma}{2}$ is the standard element for the corresponding copy of \mathfrak{sl}_2^\wedge . Using Lemma 1.3.3 we obtain

$$\hat{R}_0 e^{\hat{\rho}_\gamma} Y = (1 + \sum f_n) \mathcal{F}_{\hat{W}_\gamma} \left(\frac{\hat{R}_1 e^{\hat{\rho}_\gamma}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} \right).$$

Repeating the above reasoning for \hat{W}_γ we obtain $(\mu, \gamma) = 0$. Hence $(\mu, \alpha) = (\mu, \gamma) = 0$ and $\mu \in -\hat{Q}^+$. This implies $\mu = -m\delta + k(\beta_1 - \beta_2)$, which contradicts to the construction

of Y_2 . Hence $Y_2 = 0$ so $Y = Y_1$ that is $\text{supp}(Y) \subset \{-n\delta + j(\beta_1 - \beta_2)\}$. Combining the condition $\text{supp}(Y) \subset -\hat{Q}^+$ and (6), we obtain the required inclusion (7). \square

2.3. Evaluation. By Lemma 2.2, $\frac{RHS}{LHS}$ is a function of one variable $y := \frac{y_1}{y_2}$. In order to establish the identity $LHS = RHS$, it is enough to verify that $\frac{RHS}{LHS}(y) = 1$ for a fixed x and some y_2, y_1 satisfying $y_1 = yy_2$. We will check this for $x = -1, y_2 = y, y_1 = y^2$, (i.e. $e^{-\alpha} = -1, e^{-\beta_1} = y^2, e^{-\beta_2} = y$).

One has $\frac{RHS}{LHS} = \hat{R}^{-1}(RHS \cdot e^{-\rho})$. We write $RHS \cdot e^{-\rho} = AB$, where

$$A := \frac{((1-q)_q^\infty)^2}{(1-qq)_q^\infty(1-qq^{-1})_q^\infty}, \quad B := e^{-\rho} \cdot \mathcal{F}_{\hat{W}_\alpha}\left(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\right), \quad y = \frac{y_1}{y_2},$$

2.3.1. Recall that an infinite product $\prod_{i=1}^\infty (1+g_i(z))$, where $g_i(z)$ are holomorphic functions in $U \subset \mathbb{C}$ is called *normally convergent in U* if $\sum g_i(z)$ normally converges in U . By [R], a normally convergent infinite product converges to a function $g(z)$, which is holomorphic in U ; moreover, the set of zeros of $g(z)$ is the union of the sets of zeros of $1+f_i(z)$ and the order of each zero is the sum of the orders of the corresponding zeros of $1+g_i(z)$.

The denominator of $A(y)$ normally converges in any $U \subset X$, where $X \subset \mathbb{C}$ is a compact not containing 0. Thus $A(y)$ is a meromorphic function in the region $0 < |y|$ with simple poles at the points $y = q^n, n \in \mathbb{Z} \setminus \{0\}$.

2.3.2. The evaluation of \hat{R} takes the form

$$\hat{R}(y) = \frac{2 \prod_{n=1}^\infty (1-q^n)^4 (1+q^n)^2 (1+q^{n-1}y^3)(1+q^n y^{-3})}{\prod_{n=0}^\infty \prod_{s=1}^2 (1+q^n y^s)(1+q^{n+1} y^{-s})(1-q^n y^s)(1-q^{n+1} y^{-s})}.$$

All infinite product in the above expression normally converge in any $U \subset X$, where $X \subset \mathbb{C}$ is a compact not containing 0. Therefore $\hat{R}(y)$ is a meromorphic function in the region $0 < |y|$, and

$$(8) \quad \frac{A}{\hat{R}}(y) = \frac{(1-y) \prod_{n=0}^\infty (1-q^n y^2)(1-q^{n+1} y^{-2}) \prod_{s=1}^2 (1+q^n y^s)(1+q^{n+1} y^{-s})}{2 \prod_{n=0}^\infty (1-q^{2n})^2 (1+q^n y^3)(1+q^{n+1} y^{-3})}$$

is a meromorphic function in the region $0 < |y|$ with simple poles, the zero of order two at $y = 1$ and all other zeros of order one; the set of poles (resp., zeros) is P (resp., Z):

$$P := \{y \mid y^3 = -q^m \text{ \& } y \neq -q^k\}_{k,m \in \mathbb{Z}}, \quad Z := \{y \mid y^2 = \pm q^m\}_{m \in \mathbb{Z}}.$$

One readily sees from (8) that

$$(9) \quad \lim_{y \rightarrow 1} (y-1)^{-2} \frac{A}{\hat{R}}(y) = 2, \quad \frac{A}{\hat{R}}(qy) = \frac{A}{\hat{R}}(y) \cdot \frac{q(1-qq)}{1-y}.$$

2.3.3. Recall that

$$B = \sum_{n=-\infty}^{\infty} \left(\frac{q^n}{(1+q^n e^{-\beta_1})(1+q^n e^{-\beta_2})} - \frac{q^n e^{-\alpha}}{(1+q^n e^{-\beta_1-\alpha})(1+q^n e^{-\beta_2-\alpha})} \right)$$

so the evaluation takes the form

$$(10) \quad \begin{aligned} B(y) &= \sum_{n=-\infty}^{\infty} \left(\frac{q^n}{(1+q^n y)(1+q^n y^2)} + \frac{q^n}{(1-q^n y)(1-q^n y^2)} \right) \\ &= \frac{1}{1-y} \sum_{n=-\infty}^{\infty} \left(\frac{q^n}{1+q^n y} - \frac{q^n y}{1+q^n y^2} + \frac{q^n}{1-q^n y} - \frac{q^n y}{1-q^n y^2} \right). \end{aligned}$$

Each point $y \in \mathbb{C}$ such that $y^2 \neq \pm q^n$ for $n \in \mathbb{Z}$ has a neighborhood U such that the above sums converge absolutely and uniformly. Thus $B(y)$ is a meromorphic function in the region $0 < |y|$ with poles at the points $\{y \mid y^2 = \pm q^n\}_{n \in \mathbb{Z}}$, where all poles are simple except the pole of order two at $y = 1$. Let us verify that $B(y) = 0$ for each $y \in P$. For $y^3 = -q^k$, $y \notin \{-q^m\}$ one has

$$\frac{y}{1 \pm q^n y^2} = \frac{y}{1 \mp q^{n+k} y^{-1}} = \mp \frac{1}{1 \mp q^{-n-k} y}$$

so $B(y) = 0$. Hence $\frac{AB}{R}(y)$ is a holomorphic function in the region $0 < |y|$.

From the second formula of (10) one sees that $B(qy) = q^{-1} \frac{1-y}{1-xy}$; combining with (9) we get $\frac{AB}{R}(qy) = \frac{AB}{R}(y)$. Since $\frac{AB}{R}(y)$ is a holomorphic function in the region $0 < |y|$, this function is constant. One has

$$\lim_{y \rightarrow 1} (1-y)^2 \cdot B(y) = \lim_{y \rightarrow 1} (1-y)^2 \frac{1}{(1-y)(1-y^2)} = \frac{1}{2}.$$

Using (9) we obtain $\frac{AB}{R}(1) = 1$ so $\frac{AB}{R}(y) \equiv 1$ (for $0 < |y|$). This completes the proof of denominator identity.

3. APPLICATION TO JACOBI IDENTITY (2)

Recall the Gauss' identity (which follows easily from the Jacobi triple product)

$$\square(-q) = \frac{(1-q)_q^\infty}{(1+q)_q^\infty}.$$

The evaluation of the identity (1) at $y_1 = y_2 = 1$ gives

$$\frac{((1-x)_q^\infty (1-qx^{-1})_q^\infty)^2 ((1-q)_q^\infty)^4}{4((1+q)_q^\infty)^4 ((1+x)_q^\infty (1+qx^{-1})_q^\infty)^2} = \sum_{n=-\infty}^{\infty} a_n,$$

where $a_n := \frac{q^n}{(1+q^n)(1+q^n)} - \frac{q^n x}{(1+q^n x)(1+q^n x)}$.

We divide both sides of the above identity by $\frac{(1-x)^2}{16}$ and take the limit $x \mapsto 1$; we get

$$\left(\frac{(1-q)_q^\infty}{(1+q)_q^\infty} \right)^8 = 1 - 16 \sum_{n=1}^{\infty} \frac{q^n (q^{2n} - 4q^n + 1)}{(1+q^n)^4},$$

since

$$\lim_{x \rightarrow 1} \frac{a_0}{(x-1)^2} = \frac{1}{16}, \quad \lim_{x \rightarrow 1} \frac{a_n + a_{-n}}{(x-1)^2} = -\frac{q^n(q^{2n} - 4q^n + 1)}{(1+q^n)^4}.$$

Using the expansion $(a+1)^{-4} = \sum_{j=0}^{\infty} (-1)^j \frac{(j+1)(j+2)(j+3)}{6} a^j$, we obtain

$$\square(-q)^8 = \left(\frac{(1-q)_q^{\infty}}{(1+q)_q^{\infty}} \right)^8 = 1 + 16 \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (-1)^j j^3 q^{nj},$$

which implies the required identity

$$\square(q)^8 = 1 + 16 \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{j+nj} j^3 q^{nj}.$$

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